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Translated by Z.L.

J. Appl. Maths Mechs, Vol. 55, No. 1, pp. 48-55, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00

QUASILINEAR CONFLICT-CONTROLLED PROCESSES WITH NON-FIXED TIME*

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We identify a class of conflict-controlled processes /1-3/ for which the solving functions of the group pursuit problem /4-7/ are independent of the termination time of the game while evader errors cause the process to terminate earlier than the guarantee time. Sufficient conditions are derived for the solvability of pursuit and evasion problems, and the continuity property of the solving functions is studied in detail. The sufficient conditions for the pursuit problem to be solvable do not include Pontryagin's condition /3, 8/; it is replaced with a weaker assumption related to the initial state of the process. The proposed procedure enables us to strengthen some known results on the solution of group pursuit problems.

1. The motion of a conflict-controlled object $z = (z_1, \ldots, z_n)$ in the finite-dimensional Euclidean space R^{ν} is described by the system of differential equations

$$z_i^{\cdot} = A_i z_i + \varphi_i(u_i, v),\ z_i \in R^{\nu_i},\ u_i \in U_i,\ v \in V,\ z_i(0) = z_i^0 \tag{1.1}$$

Here A_i is a given square matrix of order ν_i, U_i and V are non-empty compact subsets in the spaces R^{p_i} and R^q, respectively, and the function $\varphi_i(u_i, v)$ is continuous in all its variables. Here and henceforth, $i = 1, 2, \ldots, n$.

The terminal set M^* consists of the sets M_i^*, such representable in the form

$$M_i^* = M_i^0 + M_i \tag{1.2}$$

where M_i^0 is a linear subspace of the space R^{ν_i} and M_i is a convex compact set in L_i - the orthogonal complement of M_i^0 in R^{ν_i}.

We say that the game (1.1) terminates from the initial state $z^0 = (z_1^0, \ldots, z_n^0)$ not later than in a time $T(z^0)$ if measurable functions $u_i(t) = u_i(z_i^0, v(t)) \in U_i$, $0 \leqslant t < t^*$, $t^* \leqslant T(z^0)$ exist such that $z_i(t^*) \in M_i^*$ for at least one i for any measurable function $v(t) \in V$, $0 \leqslant t \leqslant T(z^0)$, where $z_i(t)$ is the solution of the system of Eqs.(1.1) corresponding to the pair of controls $u_i(t)$ $v(t)$ and the initial state z_i^0.

*Prikl.Matem.Mekhan.,55,1,63-71,1991

We say that the game (1.1) from the initial state $z^0 \in R^\nu \setminus M^*$ allows evasion of the set M^* if there exists a measurable function $v(t) \in V$, $0 \leqslant t < \infty$ such that $z_i(t) \notin M_i^*$ for all $t \in [0, \infty)$ for all measurable functions $u_i(t) \in U_i$, $0 \leqslant t < \infty$. The evader applies a programmed control, i.e., a control that utilizes only information about the initial state z^0.

2. Denote by $\operatorname{int} H, \bar{H}, \partial H, \operatorname{co} H$, and $\operatorname{con} H$ respectively the interior, the closure, the boundary, the convex hull, and the conical Hull of the subset H in the space R^k; let $B_r^k(x)$ be a sphere centred at the point $x \in R^k$ with radius $r > 0$, i.e.,

$$B_r^k(x) = \{y \in R^k: \|y - x\| \leqslant r\} \tag{2.1}$$

In the space $\Omega(R^k)$ of all non-empty compact subsets of the space R^k, we define the Hausdorff metric $h(A, B)$ between two sets $A, B \in \Omega(R^k)$ by the formula

$$h(A, B) = \min\{r \geqslant 0: A \subset B + B_r^k(0), B \subset A + B_r^k(0)\} \tag{2.2}$$

In what follows we also consider the space $\operatorname{co}\Omega(R^k)$ that consists of all non-empty compact convex subsets in the space R^k.

For the set $F \in \Omega(R^k)$ we define the support function

$$c(F, \psi) = \max_{f \in F}(f, \psi), \ \psi \in R^k \tag{2.3}$$

Let $\psi_0 \in R^k$, $\|\psi_0\| \neq 0$. The set

$$U(F, \psi_0) = \{f \in F: (f, \psi_0) = c(F, \psi_0)\} \tag{2.4}$$

is called the support set to the set F in the direction ψ_0. If the support set $U(F, \psi_0)$ consists of a single point, then we say that the set F is strictly convex in the direction $\psi_0 \in R^k$ /9/. We say that the set $F \in \Omega(R^k)$ is strictly convex if it is strictly convex in every direction $\psi_0 \in R^k$, $\|\psi_0\| \neq 0$. The set $F \in \Omega(R^k)$ is called a compact set with a smooth boundary if

$$U(F, \psi) \cap U(F, \psi') = \varnothing, \ \forall \psi, \psi' \in \partial B_1^k(0), \psi \neq \psi' \tag{2.5}$$

We will state some auxiliary propositions.

Lemma 1. Let $X \in \Omega(R^p)$, $Y \in \Omega(R^q)$, $B \in \Omega(R^k)$, $\gamma: X \times Y \to \Omega(R^k)$ be an upper semicontinuous multivalued mapping, $f: X \to R^k$ a continuous function, and

$$f(e) \notin B, \ -\overline{\operatorname{con}}(f(x) - B) \cap A(x, y) \neq \varnothing, \ \forall x \in X, \ y \in Y$$

Then the function $\alpha: X \times Y \to R$, defined by the formula

$$\alpha(x, y) = \max\{\alpha \geqslant 0: -\alpha(f(x) - B) \cap A(x, y) \neq \varnothing\} \tag{2.6}$$

is upper semicontinuous.

Proof. We will show that the function $\alpha(x, y)$ is upper semicontinuous at an arbitrary point (x_0, y_0) in $X \times Y$. To this end, let

$$\alpha_0' = \limsup_{(x, y) \to (x_0, y_0)} \alpha(x, y), \ (x, y) \in X \times Y \tag{2.7}$$

Take a sequence of points $(x_r, y_r) \in X \times Y$ such that

$$\lim_{r \to \infty} \alpha(x_r, y_r) = \alpha_0'$$

Since

$$-\alpha(x_r, y_r)(f(x_r) - B) \cap A(x_r, y_r) \neq \varnothing$$

there is a vector $b_r \in B$ such that

$$-\alpha(x_r, y_r)(f(x_r) - b_r) \in A(x_r, y_r)$$

By the compactness of the set B, we isolate from the sequence $\{b_r\}$ a subsequence $\{b_{r_l}\}$ that converges to some point $b_0 \in B$. Then

$$-\alpha(x_{r_l}, y_{r_l})(f(x_{r_l}) - b_{r_l}) \in A(x_{r_l}, y_{r_l}) \tag{2.8}$$

The multivalued mapping $A(x, y)$ is upper semicontinuous, and passing to the limit as $r_l \to \infty$ we obtain from (2.8)

$$-\alpha_0'(f_0 - b_0) \in A_0 \ (f_0 = f(x_0), \ A_0 = A(x_0, y_0))$$

Using the definition of the function $\alpha(x, y)$, we obtain

$$\alpha_0' \leqslant \alpha_0 \ (\alpha_0 = \alpha(x_0, y_0))$$

Thus, the function $\alpha(x, y)$ is upper semicontinuous at the point (x_0, y_0).

Lemma 2. Let $X \in \Omega(R^p)$, $Y \in \Omega(R^q)$, $f: X \to R^k$ be a continuous function, $\|f(x)\| \neq 0$, $\forall x \in X$, and let the mulitvalued mapping $\Psi: X \to \Omega(R^k)$ be defined by the formula

$$\Psi(x) = \{\psi \in \partial B_1^k(0): (\psi, f(x)) = 0\} \tag{2.9}$$

Let $A: X \times Y \to \operatorname{co}\Omega(R^k)$ be a continuous multivalued mapping, the set $A(x, y)$ for each point $(x, y) \in X \times Y$ is strictly convex in every direction $\psi \in \Psi(x)$, and

$$-\overline{\operatorname{con}}\, f(x) \cap A(x, y) \neq \varnothing, \ \forall x \in X, \ y \in Y$$

Then the function $\alpha: X \times Y \to R$ defined by the formula

$$\alpha(x, y) = \max\{\alpha \geqslant 0: -\alpha f(x) \in A(x, y)\} \tag{2.10}$$

is continuous.

Proof. Upper semicontinuity of the function $\alpha: X \times Y \to R$ follows from Lemma 1. We will show that $\alpha(x, y)$ is lower semicontinuous. Assume that this is not so: at some point (x_0, y_0) on the set $X \times Y$ the function $\alpha(x, y)$ is not lower semicontinuous, i.e., there exists a sequence $\{(x_r, y_r)\}$, $(x_r, y_r) \in X \times Y$, that converges to the point (x_0, y_0) and

$$\lim_{r \to \infty} \alpha(x_r, y_r) = \alpha_0' < \alpha_0$$

From the definition of the function $\alpha(x, y)$, we obtain

$$-\alpha_0 f_0 \in \partial A_0, \ -\alpha(x_r, y_r) f(x_r) \in \partial A(x_r, y_r)$$

Since the continuous multivalued mapping $A(x, y)$ is convex-valued, the mapping $\partial A: X \times Y \to \Omega(R^k)$ is also continuous /10/. Therefore

$$-\alpha_0' f_0 \in \partial A_0$$

If the segment $G = [-\alpha_0' f_0, -\alpha_0 f_0]$ is on the boundary of the set A_0, then there exists a vector $\psi \in \partial B_1^k(0)$ for which $G \subset U(A_0, \psi)$ and $(f_0, \psi) = 0$, which contradicts the condition of strict convexity of the set A_0 in any direction $\psi \in \Psi(x_0)$. Thus, each point of the interval G which is not one of its ends $-\alpha_0 f_0$, $-\alpha_0' f_0$ is an interior point of A_0.

Let $\rho = \alpha_0 - \alpha_0'$. Since the sequence $\{\alpha(x_r, y_r)\}$ converges to α_0', for $\varepsilon = \rho/3$ we can identify a natural number N_1 such that

$$|\alpha(x_r, y_r) - \alpha_0'| \leqslant \varepsilon, \ \forall r \geqslant N_1$$

The point $p = -\frac{1}{3}\alpha_0' f_0 - \frac{2}{3}\alpha_0 f_0$ is in the interior of the set A_0. The multivalued mapping $A(x, y)$ is continuous, and so there exists a natural number N_2 such that for $r \geqslant N_2$ the point $p \in \operatorname{int} A(x_r, y_r)$. Thus, for $r \geqslant N_2$,

$$\alpha(x_r, y_r) \geqslant \tfrac{1}{3}\alpha_0' + \tfrac{2}{3}\alpha_0$$

Finally, for $r \geqslant \max\{N_1, N_2\}$

$$\alpha(x_r, y_r) - \alpha_0' \leqslant \tfrac{1}{3}\rho, \ \alpha_0 - \alpha(x_r, y_r) \leqslant \tfrac{1}{3}\rho \tag{2.11}$$

Adding inequalities (2.11) we obtain a contradiction,

$$\alpha_0 - \alpha_0' \leqslant \tfrac{2}{3}\rho$$

The function $\alpha: X \times Y \to R$ is thus lower semicontinuous.

Lemma 3. Let $X \in \Omega(R^p)$, $Y \in \Omega(R^q)$, $A: X \times Y \to \operatorname{co}\Omega(R^k)$ a continuous multivalued mapping, $f: X \to R^k$ a continuous function and

$$\|f(x)\| \neq 0, \ -\overline{\operatorname{con}}\, f(x) \cap \operatorname{int} A(x, y) \neq \varnothing, \ \forall x \in X, \ y \in Y. \tag{2.12}$$

Then the function $\alpha: X \times Y \to R$, defined by formula (2.10), is continuous.

The proof of Lemma 3 follows the same scheme as Lemma 2. Note only that the interval $(-\alpha_0' f_0, -\alpha_0 f_0)$ is in the interior of the set A_0 by the second relationship in (2.12).

Lemma 4. Let $A \in \operatorname{co}\Omega(R^2)$, $Y \in \Omega(R^2)$, $x \in R^2 \setminus \{0\}$ $-\overline{\operatorname{con}}\, x \cap (A - y) \neq \varnothing$, $\forall y \in Y$. Then the function $\alpha: Y \to R$ defined by the formula

$$\alpha(y) = \max\{\alpha \geqslant 0: -\alpha x \in A - y\} \tag{2.13}$$

is continuous.

Proof. We will show that the function $\alpha(y)$ is lower semicontinuous at an arbitrary point y_0 of the set Y. Assume that this is not so: there exists a sequence $\{y_r\}$, $y_r \in Y$, that converges to y_0 such that the corresponding sequence $\{\alpha(y_r)\}$ converges to α_0', $\alpha_0' < \alpha_0$. As before, we obtain that

$$-\alpha_0' x \in \partial A - y_0, \ -\alpha_0 x \in \partial A - y_0$$

If the interval $(-\alpha_0' x, -\alpha_0 x)$ is in the interior of the set $A - y_0$, then the rest of the analysis follows the same line as in the proof of Lemma 2. We therefore assume that the interval $G = [-\alpha_0' x, -\alpha_0 x]$ is on the boundary of the set $A - y_0$. Put $\rho = \alpha_0 - \alpha_0'$. Since $\alpha(y_r) \to \alpha_0'$ as $r \to \infty$, we may assume that for any $r \geqslant 1$,

$$\| y_r - y_0 \| < {}^1/_4\rho \| x \|, \ | \alpha (y_r) - \alpha_0' | < {}^1/_4\rho \tag{2.14}$$

Since $G \subset \partial A - y_0$, there is a vector $\psi_0 \in \partial B_1^2(0)$ such that $G + y_0 \in U(A, \psi_0)$. Hence we obtain that $(x, \psi_0) = 0$ and $c(A, \psi_0) = (y_0, \psi_0)$. Since $-\alpha(y_r)x + y_r \in A$, we have $(y_r, \psi_0) \leqslant (y_0, \psi_0)$.

We will show that $(y_r, \psi_0) < (y_0, \psi_0)$ for any $r \geqslant 1$. Indeed,

$$-{}^1/_2(\alpha_0 + \alpha_0')x + y_0 - y_r + y_r \in A$$

Therefore, if $(y_r, \psi_0) = (y_0, \psi_0)$, then $y_0 - y_r = \alpha x$, $|\alpha| < {}^1/_4\rho$ and therefore $\alpha(y_r) > \alpha_0' + {}^1/_4\rho$, which contradicts (2.14). Thus, for any $r \geqslant 1$ the point $-\alpha(y_r)x + y_r$ lies on the line through the points $-\alpha_0'x + y_0$, $-\alpha_0 x + y_0$. Consider the sets

$$K = \mathrm{co}\{-\alpha_0'x + y_0, -\alpha_0 x + y_0, -\alpha(y_1)x + y_1\}$$
$$K_\delta = \{z \in R^2: z \in B_\delta^2(-{}^1/_2(\alpha_0 + \alpha_0')x + y_0), (z, \psi_0) \leqslant (y, \psi_0)\}$$

Clearly, $K \subset A$ and there exists $\delta > 0$ for which $K_\delta \subset K$. Starting with some N, the elements of the sequence $\{z_r\}$, $z_r = -{}^1/_2(\alpha_0 + \alpha_0')x + y_r$ are in the set K_δ. At the same time, from conditions (2.14) we have

$$[-(\alpha_0' + {}^1/_4\rho)x, -\alpha_0 x] + y_r \cap A = \varnothing, \ r = 1, 2, \ldots$$

Therefore, the set A does not contain any elements from the sequence $\{z_r\}$. A contradiction. The function $\alpha(y)$ is thus lower continuous on the set Y.

Remark 1. Already in the space R^3 with all other assumptions of Lemma 4 satisfied, the function $\alpha(y)$ defined by the formula (2.13) is not necessarily lower semicontinuous, even if $Y = A$, $0 \in A$.

3. Let us describe the pursuit scheme. Denote by π_i the orthogonal projector from R^{ν_i} on L_i.

Condition 1. For a fixed point $z = (z_1, \ldots, z_n) \in R^\nu$ such that $\pi_i \exp(tA_i) \notin M_i$ for $t \geqslant 0$, we have the relationships

$$\overline{-\mathrm{con}}\,(\pi_i \exp(tA_i) z_i - M_i) \cap \pi_i \exp((t - \tau) A_i) \varphi_i (U_i, v) \neq \varnothing \tag{3.1}$$

for all $0 \leqslant \tau \leqslant t < \infty$, $v \in V$.

Fix a point z for which Condition 1 is satisfied and introduce the solving functions

$$\alpha_i(z_i, t, \tau, v) = \max\{\alpha \geqslant 0: -\alpha(\pi_i \exp(tA_i) z_i - M_i) \cap$$
$$\cap \pi_i \exp((t - \tau) A_i) \varphi_i (U_i, v) \neq \varnothing\}, \ 0 \leqslant \tau \leqslant t < \infty, \ v \in V \tag{3.2}$$

Corollary. Assume that the point $z = (z_1, \ldots, z_n) \in R^\nu$ satisfies Condition 1. Then for any $T \in [0, \infty)$ the function $\alpha_i(z_i, t, \tau, v)$: $[0, T] \times [0, T] \times V \to R$ is upper semicontinuous.

Proof. The function φ_i: $U_i \times V \to R^{\nu_i}$ is continuous, and therefore the multivalued mapping $\varphi_i(U_i, v)$: $V \to \Omega(R^{\nu_i})$ is continuous /11/. Since the matrix $\exp((t - \tau)A_i)$ continuously depends on (t, τ), the multivalued mapping $\pi_i \exp((t - \tau)A_i)\varphi_i(U_i, v)$: $[0, T] \times [0, T] \times V \to \Omega(R^{\nu_i})$ is continuous. Lemma 1 proves the corollary.

Put

$$\lambda(z, t) = 1 - \inf_{v_t(\cdot)} \max_i \int_0^t \alpha_i(z_i, t, \tau, v(\tau))\, d\tau \tag{3.3}$$
$$T(z) = \inf\{t \geqslant 0: \lambda(z, t) \leqslant 0\} \tag{3.4}$$

The infimum in (3.3) is over all possible functions measurable in $[0, t]$ with values in the set V.

Theorem 1. Let $M_i = \{0\}$, $\pi_i A_i = A_i \pi_i$, the point $z^0 = (z_1^0, \ldots, z_n^0) \in R^\nu$ satisfies Condition 1 and $T^0 = T(z^0) < \infty$.

Then the game (1.1) will terminate from the initial state z^0 not later than in time T^0.

Proof. From the assumptions of the theorem it follows that the functions $\alpha_i(z_i^0, t, \tau, v)$ are independent of t and can be represented in the form

$$\alpha_i(z_i^0, \tau, v) = \max\{\alpha \geqslant 0: -\alpha\pi_i z_i^0 \in \pi_i \exp(-\tau A_i) \varphi_i(U_i, v)\}$$
$$z_i^0 \notin M_i^*, \ 0 \leqslant \tau < \infty, \ v \in V$$

Let $v(\tau)$, $0 \leqslant \tau \leqslant T^0$, be some measurable function with values in the set V. Since the function $\alpha_i(z_i^0, \tau, v)$: $[0, T^0] \times V \to R$ is upper semicontinuous, the function $\alpha_i(z_i^0, \tau, v(\tau))$ is measurable with respect to τ in the interval $[0, T^0]$. Let

$$h(t) = 1 - \max_i \int_0^t \alpha_i(z_i^\circ, \tau, v(\tau))\, d\tau$$

Let t_* be the first positive root of the equation $h(t) = 0$. This root exists and $t_* \leqslant T^0$. This follows from the continuity of the function $h(t)$ and the inequality $h(T^0) \leqslant 0$.

By Condition 1 and the Filippov-Kasten theorem /12/, for any i there exists a measurable function $u_i(\tau) \in U_i$, $0 \leqslant \tau \leqslant T^0$, which for any fixed $\tau_0 \in [0, T^0]$ solves the equation

$$-\alpha_i(z_i^0, \tau_0, v(\tau_0))\, \pi_i z_i^0 = \pi_i \exp(-\tau_0 A_i)\, \varphi_i(u_i, v(\tau_0)) \tag{3.5}$$

At time τ_0, the value of the control $u_i(\tau_0)$ is a lexicographic minimum among all the points $u_i \in U_i$ for which equality (3.5) holds.

We will show that if the pursuer controls are chosen in this way, then $z(t_*) \in M^*$. Indeed, at time t_* there is an index $s \in \{1, \ldots, n\}$, for which

$$1 - \int_0^{t_*} \alpha_s(z_s^\circ, \tau, v(\tau))\, d\tau = 0$$

The matrices π_s and A_s commute, and therefore by the Cauchy formula we have

$$\pi_s z_s(t_*) = \exp(t_* A_s)\left(\pi_s z_s^\circ + \int_0^{t_*} \pi_s \exp(-\tau A_s)\, \varphi_s(u_s(\tau), v(\tau))\, d\tau\right) =$$

$$\exp(t_* A_s)\left(\pi_s z_s^\circ - \pi_s z_s^\circ \int_0^{t_*} \alpha_s(z_s^\circ, \tau, v(\tau))\, d\tau\right) = 0.$$

Hence, $z_s(t_*) \in M_s^*$.

4. Let us consider in more detail the conflict-controlled process (1.1) in the case when A_i is the zero square matrix of order ν_i for any i. The conflict-controlled process of the type of simple motion with mixed player controls is described by the system of differential equations

$$z_i^{\cdot} = \varphi_i(u_i, v),\ z_i \in R^{\nu_i}, u_i \in U_i,\ v \in V,\ z_i(0) = z_i^0 \tag{4.1}$$

Here $U_i \in \Omega(R^{p_i})$, $V \in \Omega(R^q)$, the function $\varphi_i(u_i, v)$ is continuous in all the variables. The terminal set M^* consists of the sets M_i^*, each representable in the form (1.2).

Form the multivalued mappings

$$W_i(z_i, v) = -\overline{\text{con}}\,(\pi_i z_i - M_i) \cap \pi_i \varphi_i(U_i, v)$$

$$\overline{W}_i(z_i, v) = -\overline{\text{con}}\,(\pi_i z_i - M_i) \cap \text{co}\, \pi_i \varphi_i(U_i, v),\ z_i \in R^{\nu_i} \setminus M_i^*, v \in V$$

Condition 2. For a fixed point $z = (z_1, \ldots, z_n) \in R^\nu \setminus M^*$, we have $W_i(z_i, v) \neq \varnothing$ for any $v \in V$.

Condition 3. For a fixed point $z = (z_1, \ldots, z_n) \in R^\nu \setminus M^*$, we have $\overline{W}_i(z_i, v) \neq \varnothing$ for any $v \in V$.

Fix the points $z, \bar{z}$ for which Conditions 1 and 2 hold respectively and introduce the solving functions

$$\alpha_i(z_i, v) = \max\{\alpha \geqslant 0: -\alpha(\pi_i z_i - M_i) \cap \pi_i \varphi_i(U_i, v) \neq \varnothing\} \tag{4.2}$$

$$\bar{\alpha}_i(z_i, v) = \max\{\alpha \geqslant 0: -\alpha(\pi_i \bar{z}_i - M_i) \cap \text{co}\, \pi_i \varphi_i(U_i, v) \neq \varnothing\},\ v \in V \tag{4.3}$$

Consider the function

$$T(z) = \inf\left\{t \geqslant 0: 1 - \inf_{v_t(\cdot)} \max_i \int_0^t \alpha_i(z_i, v(\tau))\, d\tau \leqslant 0\right\}$$

Let

$$\alpha(z) = \inf_{v \in V} \max_i \alpha_i(z_i, v), \quad \bar{\alpha}(\bar{z}) = \inf_{v \in V} \max_i \bar{\alpha}_i(\bar{z}_i, v)$$

Theorem 2. Assume that the point $z^0 = (z_1^0, \ldots, z_n^0) \in R^\nu \setminus M^*$ satisfies Condition 2 and $\alpha^0 > 0$. Then the game (4.1) will terminate from the initial state z^0 not later than in time T° with the bound $T^\circ \leqslant n/\alpha^\circ$.

The proof is based on the ideas used in the proof of Theorem 1. Let us derive an upper bound of T°. Since

$$1 - \inf_{v_t(\cdot)} \max_i \int_0^t \alpha_i(z_i^\circ, v(\tau))\, d\tau \leqslant 1 - \frac{1}{n} \inf_{v_t(\cdot)} \int_0^t \sum_i \alpha_i(z_i^\circ, v(\tau))\, d\tau \leqslant$$
$$1 - \frac{1}{n} \inf_{v_t(\cdot)} \int_0^t \max_i \alpha_i(z_i^\circ, v(\tau))\, d\tau = 1 - \frac{1}{n}\alpha(z^\circ)\, t$$

we have $T^\circ \leqslant n/\alpha^\circ$ if $\alpha^\circ > 0$.

Theorem 3. Assume that the point $z^\circ = (z_1^\circ, \ldots, z_n^\circ) \in R^\nu \setminus M^*$ satisfies Condition 3, $\bar{\alpha}(z^\circ) = 0$, and the infimum in the expression

$$\inf_{v \in V} \max_i \bar{\alpha}_i(z_i^\circ, v)$$

is attained on some vector $v_0 \in V$. Then the game (4.1) from the initial state z° allows evasion of the set M^*.

Proof. Let $v(t) \equiv v_0$ for $t \geqslant 0$. Since $\bar{\alpha}(z^\circ) = 0$, then $\bar{\alpha}_i(z_i^\circ, v_0) = 0$. This in turn implies that

$$-\mathrm{con}\,(\pi_i z_i^\circ - M_i) \cap \mathrm{co}\, \pi_i \varphi_i(U_i, v_0) = \varnothing$$

Hence we obtain

$$\{\pi_i z_i^\circ + t\, \mathrm{co}\, \pi_i \varphi_i(U_i, v_0)\} \cap M_i = \varnothing, \ \forall t > 0 \tag{4.4}$$

Note that

$$\pi_i z_i(t) \subset \pi_i z_i^\circ + t\, \mathrm{co}\, \pi_i \varphi_i(U_i, v_0), \ \forall t > 0$$

Therefore, using (4.4), we obtain that $\pi_i z_i(t) \notin M_i$ for $t > 0$.

Remark 2. If $M_i = \{0\}$, then Lemmas 2-4 supply a sufficient condition for the infimum in the expression

$$\inf_{v \in V} \max_i \alpha_i(z_i^\circ, v)$$

to be attained.

In general (game (1.1)), the assumptions $\pi_i A_i = A_i \pi_i$, $M_i = \{0\}$ limit the solvable class of pursuit problems, but the conditions on the pursuer control regions in the proposed scheme are in some sense weaker than in standard methods of solving the pursuit problem.

Let us demonstrate this result with some examples.

Example 1. We are given the differential game

$$z_i^{\cdot} = u_i - v,\ z_i \in R^k,\ u_i \in U_i(z_i^0),\ v \in V,\ z_i(0) = z_i^0$$

Here V is a strictly concave and convex compact set with a smooth boundary, $U_i(z_i^\circ) = \bigcup_{\psi \in S^-(z_i^\circ)} U(V, \psi)$, $S^-(z_i^\circ) = \{\psi \in \partial B_1^K(0) \colon (\psi, z_i^\circ) \leqslant 0\}$ and $M_k^* = \{0\}$

A necessary condition for the applicability of the known methods of group pursuit /4-7, 13, 14/ to solve the pursuit problem in this differential game is the condition

$$\bigcap_{v \in V} (U_i - v) \neq \varnothing$$

which is not satisfied in this case, because the control region of each pursuer is only part of the boundary of the control region of the evader.

It can be shown that the point $z^\circ = (z_1^\circ, \ldots, z_n^\circ) \in R^{kn} \setminus M^*$ satisfies Condition 2 and $\alpha^\circ > 0$ if and only if we have the inclusion

$$0 \in \operatorname{int} \operatorname{co} \{z_1^{*}, \ldots, z_n^{*}\} \tag{4.5}$$

Example 2. The differential game is defined by the system of equations

$$z_i^{\cdot} = u_i - v,\ z_i \in R^k,\ u_i \in \partial B_1^k(0),\ v \in B_1^k(0),\ z_i(0) = z_i^0$$

As in example 1, $M_i^* = \{0\}$.

From Theorems 2 and 3 we conclude that inclusion (4.5) is a necessary and sufficient condition for terminating the game in a finite time.

5. Let $n = 1$, then the conflict-controlled process (4.1) is defined by the equation

$$z^{\cdot} = \varphi(u, v),\ z \in R^{\nu},\ u \in U,\ v \in V,\ z(0) = z^{\circ} \tag{5.1}$$

Here $U \in \Omega(R^p)$, $V \in \Omega(R^q)$, the function $\varphi(u, v)$ is continuous in all variables. The terminal set M^* is representable in the form $M^* = M^{\circ} + M$, where M° is a linear subspace of the space R^{ν} and M is a convex compact set in the orthogonal complement of L to M° in the space R^{ν}.

For the point z satisfying Condition 2 with $n = 1$ defined by formula (4.2) the function $\alpha(z, v)$ (the index i may be omitted in this case). Consider the multivalued mappings

$$W_0(z, v) = -\operatorname{con}(\pi z - M) \cap \pi\varphi(U, v)$$
$$\overline{W}_0(z, v) = -\operatorname{con}(\pi z - M) \cap \operatorname{co} \pi\varphi(U, v),\ z \in R^{\nu} \setminus M^*,\ v \in V$$

Form the following sets:

$$W = \{z \in R^{\nu} \setminus M^* : W_0(z, v) \neq \varnothing,\ \forall v \in V\}$$
$$\overline{W} = \{z \in R^{\nu} \setminus M^* : \overline{W}_0(z, v) \neq \varnothing,\ \forall v \in V\}$$

Proposition 1. Let the point $z^{\circ} \in W$, $\pi\varphi(U, v)$: $V \to \operatorname{co} \Omega(R^{\nu})$, $M = \{0\}$ and assume that at least one of the following conditions is satisfied:

1) the set $\pi\varphi(U, v)$ for each $v \in V$ is strictly convex in any direction $\psi_0 \in \Psi$, $\Psi = \{\psi \in \partial B_1^k(0)$: $(\psi, \pi z^{\circ}) = 0\}$;

2) $-\operatorname{con} \pi z^{\circ} \cap \inf \pi\varphi(U, v) \neq \varnothing$, $\forall v \in V$;

3) $\nu = 2$, $\varphi(u, v) = u - v$.

Then the game (5.1) will terminate from the initial state z° not later than in time $T^{\circ} = 1/\alpha^{\circ}$, where $\alpha^{\circ} = \min\limits_{v \in V} \alpha(z^{\circ}, v)$.

Proof. Since $z^{\circ} \in W$, the point z° satisfies Condition 2 and $\alpha(z^{\circ}, v) > 0$ for any $v \in V$. If any of the three conditions stated in the body of the proposition is satisfied, the function $\alpha(z^{\circ}, v)$ is continuous in v (Lemmas 2-4) and therefore $\alpha^{\circ} > 0$. By Theorem 2, the game (5.1) will terminate from the initial state z° not later than in time

$$T^0 = \min\left\{t \geqslant 0:\ 1 - \inf_{r_t(\cdot)} \int_0^t \alpha(z^0, v(\tau))\, d\tau = 0\right\} = \min\{t \geqslant 0:\ 1 - \alpha^0 t = 0\}$$

Hence $T^{\circ} = 1/\alpha^{\circ}$. The proposition is proved.

Proposition 2. Let the point $z^{\circ} \in R^{\nu} \setminus \overline{W}$, $z^{\circ} \notin M^*$. Then the game (5.1) from initial state z° allows evasion of the set M^*.

Proof. When the conditions of the proposition are satisfied, there is a vector $v_0 \in V$ such that

$$-\operatorname{con}(\pi z^{\circ} - M) \cap \operatorname{co} \pi\varphi(U, v_0) = \varnothing$$

Let $v(t) \equiv v_0$ for $t \geqslant 0$. Repeating the argument in the proof of Theorem 3, we show that for the evader control chosen in this way $z(t) \notin M^*$ for $t \geqslant 0$.

Let us emphasize some advantages of the proposed scheme. Pontryagin's condition for the game (5.1) has the form

$$K = \bigcap_{v \in V} \pi\varphi(U, v) \neq \varnothing \tag{5.2}$$

Define the function

$$P(z) = \min\{t \geqslant 0:\ \pi z \in M - t \operatorname{co} K\}$$

Theorem 4. Assume that in the game (5.1) $z^\circ \in R^\nu \setminus M^*$, condition (5.2) is satisfied, $P(z^\circ) < \infty$, and the set K is convex. Then Condition 2 is satisfied for the point z°, $\alpha^\circ > 0$, and $T^\circ \leqslant P^\circ$ $(P^\circ = P(z^\circ))$.

Proof. Since $P^\circ < \infty$, at time $t = P^\circ$ we have the inclusion $\pi z^\circ \in M - P^\circ K$, which is equivalent to the following relationship:

$$(M - \pi z^\circ) \cap P^\circ K \neq \varnothing$$

From $P^\circ > 0$ we obtain

$$1/P^\circ (M - \pi z^\circ) \cap \pi\varphi(U, v) \neq \varnothing, \ \forall v \in V$$

i.e., for the point z° Condition 2 is satisfied and $\alpha(z^\circ, v) \geqslant 1/P^\circ$, $\forall v \in V$. Therefore

$$\alpha^\circ = \inf_{v \in V} \alpha(z^\circ, v) \geqslant 1/P^\circ \tag{5.3}$$

and so $\alpha^\circ > 0$. From the inequality in (5.3) we have

$$T^\circ = 1/\alpha^\circ \leqslant P^\circ$$

Example 3. The conflict-controlled process is defined by the equation

$$z^\cdot = u - v, \ z \in R^2, u \in U, \ v \in V, \ z(0) = z^\bullet \tag{5.4}$$

The control regions are $U = \{u : u = (u_1, u_2), -1 \leqslant u_1 \leqslant 1, \ u_2 = 2\}$, $V = B_1^2(0)$. The terminal set is $M^* = \{0\}$.

Condition (5.2) does not hold for the given control regions, and therefore Pontryagin's first direct method for solving the pursuit problem does not apply. At the same time by Proposition 1, if $z^\bullet \in W = \mathrm{con}\{(0, -1)\}$, then the game (5.4) will terminate from the initial state $z^\bullet = (z^{\bullet 1}, z^{\bullet 2})$ not later than in time $T^\bullet = -z^{\bullet 2}$.

Clearly, $W = \overline{W}$. By Proposition 2 we conclude that if $z^\circ \in R^2 \setminus W$, $\|z^\circ\| \neq 0$, then game (5.4) from initial state z° allows evasion of the set M^*.

Remark 3. All the theorems and propositions of Sects.4 and 5 can be proved for the game (1.1) with $A_i = a_i E$, $a_i \in R$, $a_i < 0$, E_i the identity matrix of order ν_i, and $M_i = \{0\}$.

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Translated by Z.L.